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The Degree of Approximation to Periodic Functions by Linear Positive Operators

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1. P. P. Korovkin [*1*] has recently proved some remarkable results concerning the convergence of sequences $(L_n f)_{n=1}^{\infty}$, where the L_n are linear positive operators. For example, if $L_n f$ converges uniformly to f in the particular cases $f(t) \equiv 1, f(t) \equiv t, f(t) \equiv t^2$, then it does so for every continuous, real f. Or, if $L_n(f)$ converges uniformly to f for $f(t) \equiv 1$, $\cos t$, $\sin t$, it does so for every continuous, 2π-periodic, real f.

2. In a very recent paper [*2*], the authors have recast Korovkin's results in a quantitative form. One of their results (Theorem 3 of [*2*]) was given there as, essentially, a special case of a more general theorem. In the present note, we shall restate this Theorem 3 and, for the reader's convenience, give its full proof. We then apply it to an important special case.

3. A linear positive operator is a function L having the following properties.

a. The domain D of L is a nonempty set of real functions, all having the same real domain T.

b. For every $f \in D, L(f)$ is again a real function with domain T.

c. If f and g belong to D, and if a and b are reals, then $af + bg \in D$, and

$$L(af + bg) = aL(f) + bL(g).$$

d. If $f \in D$, and $f(x) \geqslant 0$ for every $x \in T$, then $(Lf)(x) \geqslant 0$ for every $x \in T$.

Consequently, if L is a linear positive operator and f, $g \in D$, then $f \leqslant g$ throughout T implies $Lf \leqslant Lg$ there, and $|f| \leqslant g$ throughout T implies $|Lf| \leqslant Lg$ there.

4. THEOREM [*2*]. *Let $L_1, L_2, \ldots$ be linear positive operators, whose common domain D consists of real functions with domain $(-\infty, \infty)$. Suppose* 1, $\cos x$, $\sin x$, f *belong to D, where f is an everywhere continuous, 2π-periodic function, with modulus of continuity ω. Let $-\infty < a < b < \infty$, and suppose that for $n = 1$, $2, \ldots, L_n(1)$ is bounded in $[a,b]$. Then for $n = 1, 2, \ldots$,*

$$\|f - L_n f\| \leqslant \|f\| \cdot \|L_n(1) - 1\| + \|L_n(1) + 1\| \omega(\mu_n), \tag{1}$$

where (see Remark b)

$$\mu_n = \pi \left\| \left(L_n \sin^2 \frac{t-x}{2} \right)(x) \right\|^{1/2}, \tag{2}$$

and $\| \ \|$ *stands for the sup norm over* $[a,b]$. *In particular, if* $L_n(1) = 1$, *as is often the case,* (1) *reduces to*

$$\|f - L_n f\| \leqslant 2\omega(\mu_n). \tag{3}$$

Remarks. a. In forming $L_n \sin^2 [(t-x)/2]$ in (2) and below, t is the variable.

b. Observe that (2) implies, for $n = 1, 2, \ldots,$

$$\mu_n{}^2 \leqslant (\pi^2/2)[\|1 - L_n(1)\| + \|\cos x\| \cdot \|\cos x - (L_n \cos t)(x)\| + \|\sin x\| \cdot \|\sin x - (L_n \sin t)(x)\|].$$

Hence, if $L_n(F)$ converges uniformly to F in $[a,b]$ for $F(t) \equiv F_0(t) \equiv 1$, $F(t) \equiv F_1(t) \equiv \cos t$, $F(t) \equiv F_2(t) \equiv \sin t$, then $\mu_n \to 0$ and we have a simple estimate of μ_n in terms of $\|F_k - L_n F_k\|$, $k = 0, 1, 2$.

Proof of the Theorem. Let $x \in [a,b]$, let δ be a positive number and let t be real. If $\delta < |t - x| \leqslant \pi$, then $|t-x| \leqslant \pi \sin [|t-x|/2]$ and therefore

$$\begin{aligned} |f(t) - f(x)| &\leqslant \omega(|t-x|) = \omega(|t-x|\delta^{-1}\delta) \\ &\leqslant (1 + |t-x|\delta^{-1})\omega(\delta) \\ &\leqslant [1 + (t-x)^2 \delta^{-2}]\,\omega(\delta) \\ &\leqslant \left[1 + (\pi/\delta)^2 \sin^2 \frac{t-x}{2}\right] \omega(\delta). \end{aligned}$$

The resulting inequality

$$|f(t) - f(x)| \leqslant \left[1 + (\pi/\delta)^2 \sin^2 \frac{t-x}{2}\right] \omega(\delta) \tag{4}$$

holds, obviously, if $|t-x| \leqslant \delta$. If $|t - x| > \pi$, let k be an integer such that $|(t + 2k\pi) - x| \leqslant \pi$; then

$$\begin{aligned} |f(t) - f(x)| = |f(t + 2k\pi) - f(x)| &\leqslant \left[1 + (\pi/\delta)^2 \sin^2 \frac{t + 2k\pi - x}{2}\right] \omega(\delta) \\ &= \left[1 + (\pi/\delta)^2 \sin^2 \frac{t-x}{2}\right] \omega(\delta). \end{aligned}$$

Thus, (4) always holds. Let n be a positive integer. Then

$$\begin{aligned} |[L_n f - f(x) L_n(1)](x)| &\leqslant \left[\left(L_n(1) + \delta^{-2}\pi^2 L_n \sin^2 \frac{t-x}{2}\right)(x)\right] \omega(\delta) \\ &\leqslant [L_n(1)(x) + (\mu_n/\delta)^2]\,\omega(\delta). \end{aligned}$$

If $\mu_n > 0$, take $\delta = \mu_n$. Then

$$\begin{aligned} |[L_n f - f(x) L_n(1)](x)| &\leqslant \|L_n(1) + 1\| \, \omega(\mu_n), \\ |-f(x) + f(x) L_n(1)(x)| &\leqslant \|f\| \cdot \|L_n(1) - 1\|. \end{aligned} \tag{5}$$

Adding, we obtain (1). If $\mu_n = 0$, we have for every positive δ, $|[L_n f - f(x) L_n(1)](x)| \leqslant \omega(\delta) L_n(1)(x)$. Letting $\delta \to 0+0$, we obtain $(L_n f)(x) = f(x) L_n(1)(x)$. Thus, by (5), $|(f - L_n f)(x)| \leqslant \|f\| \cdot \|L_n(1) - 1\|$, which implies (1).

5. Let D be the set of all real functions with domain $(-\infty, \infty)$, 2π-periodic and everywhere continuous. For $n = 1, 2, \ldots$, let $\rho_1^{(n)}, \rho_2^{(n)}, \ldots, \rho_n^{(n)}$ be given reals, and consider the operator L_n with domain D, defined by

$$(L_n \phi)(x) \equiv \frac{a_0}{2} + \sum_{k=1}^{n} \rho_k^{(n)} [a_k \cos(kx) + b_k \sin(kx)], \tag{6}$$

where

$$\phi(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

Assume that for $n = 1, 2, \ldots$ and every real x,

$$\frac{1}{2} + \sum_{k=1}^{n} \rho_k^{(n)} \cos(kx) \geqslant 0. \tag{7}$$

Since for $n = 1, 2, \ldots$ and every $\phi \in D$,

$$(L_n \phi)(x) \equiv \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) \left[\frac{1}{2} + \sum_{k=1}^{n} \rho_k^{(n)} \cos\{k(t-x)\}\right] dt, \tag{8}$$

each L_n is a linear positive operator with $L_n(1) = 1$. Also, for $n = 1, 2, \ldots$, we have

$$\left(L_n \sin^2 \frac{t-x}{2}\right)(x) \equiv \frac{1}{2}\left(1 - \rho_1^{(n)}\right).$$

Let $f \in D$ have modulus of continuity ω. Setting $\sigma_n(x) \equiv (L_n f)(x)$, we have by (3),

$$\max_{-\infty < x < \infty} |f(x) - \sigma_n(x)| \leqslant 2\omega(\pi[2^{-1}(1 - \rho_1^{(n)})]^{1/2}), \qquad n = 1, 2, \ldots, \tag{9}$$

and in particular, $\sigma_n(x)$ converges uniformly to $f(x)$ in $(-\infty, \infty)$ if $\rho_1^{(n)} \to 1$.

The uniform convergence of $\sigma_n(x)$ to $f(x)$ in $(-\infty, \infty)$ under the condition $\rho_1^{(n)} \to 1$ was proved by P. P. Korovkin ([*1*], [*3*]). He has also shown [*1*] that for $n = 1, 2, \ldots$ and for every positive δ,

$$\max_{-\infty < x < \infty} |f(x) - \sigma_n(x)| \leqslant \omega(\delta)\{1 + \pi\delta^{-1}[2^{-1}(1 - \rho_1^{(n)})]^{1/2}\}. \tag{10}$$

For $n = 1, 2, \ldots$, let

$$M_n = \inf_{\delta>0} \omega(\delta)\{1 + \pi\delta^{-1}[2^{-1}(1-\rho_1^{(n)})]^{1/2}\}, \tag{11}$$

so that the best estimate derivable from (10) is

$$\max_{-\infty<x<\infty} |f(x) - \sigma_n(x)| \leqslant M_n. \tag{12}$$

We show now that (12) is essentially the same estimate as (9). We start by observing that

$$\omega([1-\rho_1^{(n)}]^{1/2}) \leqslant M_n \leqslant 2\omega(\pi[2^{-1}(1-\rho_1^{(n)})]^{1/2}), \qquad n = 1, 2, \ldots. \tag{13}$$

Indeed, let n be a positive integer. To prove the last two inequalities, we may assume $1 - \rho_1^{(n)} > 0$. The right inequality in (13) is obtained from (11) by taking $\delta = \pi[2^{-1}(1-\rho_1^{(n)})]^{1/2}$. To prove the left inequality of (13), we shall show that for every $\delta > 0$,

$$\omega([1-\rho_1^{(n)}]^{1/2}) \leqslant \omega(\delta)\{1 + \pi\delta^{-1}[2^{-1}(1-\rho_1^{(n)})]^{1/2}\}.$$

We may clearly assume $\delta < (1-\rho_1^{(n)})^{1/2}$. Then, $\omega([1-\rho_1^{(n)}]^{1/2}) = \omega([1-\rho_1^{(n)}]^{1/2}\delta^{-1}\delta) \leqslant [1 + (1-\rho_1^{(n)})^{1/2}\delta^{-1}]\ \omega(\delta) \leqslant 2\delta^{-1}(1-\rho_1^{(n)})^{1/2}\omega(\delta)$. So, $\omega(\delta)\{1 + \pi\delta^{-1}[2^{-1}(1-\rho_1^{(n)})]^{1/2}\} \geqslant \omega(\delta) + 2^{-3/2}\pi\omega([1-\rho_1^{(n)}]^{1/2}) \geqslant \omega([1-\rho_1^{(n)}]^{1/2})$.

From (13) it follows that for every positive K and for $n = 1, 2, \ldots$,

$$\frac{1}{K+1}\omega(K[1-\rho_1^{(n)}]^{1/2}) \leqslant \omega([1-\rho_1^{(n)}]^{1/2}) \leqslant M_n \leqslant 2\omega\left(\frac{\pi}{K\sqrt{2}}K[1-\rho_1^{(n)}]^{1/2}\right)$$
$$\leqslant 2\left[1 + \frac{\pi}{K\sqrt{2}}\right]\omega(K[1-\rho_1^{(n)}]^{1/2}).$$

Thus, for every positive K, the sequences M_n and $\omega(K[1-\rho_1^{(n)}]^{1/2})$ are of the same order of magnitude. In particular, (9) and (12) are essentially the same estimate. Also, if the left-hand side of (10) is positive for $n = 1, 2, \ldots$, then the choice $\delta = K(1-\rho_1^{(n)})^{1/2}$ in the right-hand side of (10), $n = 1, 2, \ldots$, where K is any positive constant, can be considered an optimal choice. Taking $K = \pi/\sqrt{2}$, the resulting inequalities (10) reduce to (9).

6. *Example.* Let D be as in the first sentence of Section 5. For $n = 1, 2, \ldots$, consider the operator L_n with domain D, defined by

$$(L_n\phi)(x) \equiv \frac{a_0}{2} + \sum_{k=1}^{n} \frac{(n!)^2}{(n-k)!(n+k)!}[a_k\cos(kx) + b_k\sin(kx)],$$

where

$$\phi(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) + b_k\sin(kx).$$

For $n = 1, 2, \ldots$, the $(L_n\phi)(x)$ are trigonometric polynomials introduced by de la Vallée-Poussin [*4*]. They have the following representation:

$$(L_n\phi)(x) \equiv (n!)^2\,[2\pi(2n)!]^{-1}\int_{-\pi}^{\pi}\phi(t)\left(2\cos\frac{t-x}{2}\right)^{2n}dt. \tag{14}$$

Thus, for $n = 1, 2, \ldots, L_n\phi$ is of the form (6), and as is seen by comparing, for the present case, (8) with (14), (7) holds for every real x. Let $f \in D$ have modulus of continuity ω, and set $\sigma_n(x) \equiv (L_n f)(x)$. Since now $\rho_1^{(n)} = n/(n+1)$, $n = 1, 2, \ldots$, we have by (9),

$$\max_{-\infty<x<\infty}|f(x) - \sigma_n(x)| \leqslant 2\omega\left(\frac{\pi}{[2(n+1)]^{1/2}}\right).$$

Thus, we have obtained the (known) result ([*5*], [*6*]), that for some universal constant C,

$$\max_{-\infty<x<\infty}|f(x) - \sigma_n(x)| \leqslant C\omega(n^{-1/2}) \qquad (n = 1, 2, \ldots).$$

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